Indukované zobrazení, diffeomorfismy, toky a Lieova derivace

Indukované zobrazení na tenzorech

Diferenciál zobrazení

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Indukované zobrazení na tenzorech

připomínka:

hladké zobr $\phi : M \to N$

souř. vyjádření hladké $\bar{\phi} = y \circ \phi \circ x^{-1}$ hladké

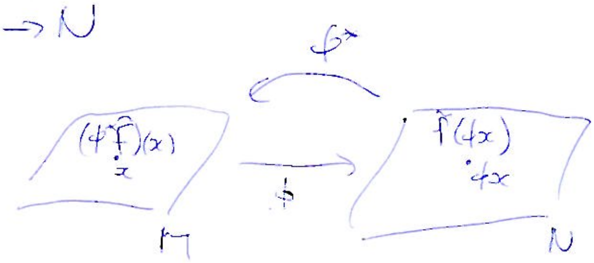
kde (U, x) mapa na M a (V, y) mapa na N

difeomorfismus $\phi : M \to N$ či $\phi : M \to M$

existuje hladké inverz. zobr ϕ^{-1}

indukované zobr. pro $\phi : M \to N$

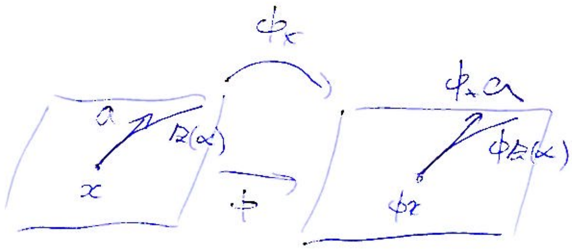
pull-back (stáhnutí) fce



$\phi^* : \mathcal{F}N \to \mathcal{F}M \qquad \hat{f} \to \phi^* \hat{f}$

$(\phi^* \hat{f})(x) = \hat{f}(\phi x)$

push-forward (přenos) vektorů

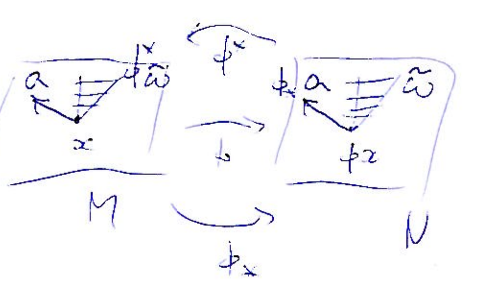


$\phi_* : \mathbb{T}_x M \to \mathbb{T}_{\phi x} N$

$$\phi_* \frac{Dz}{d\alpha}\Big|_{\alpha=0} = \frac{D \phi z}{d\alpha}\Big|_{\alpha=0} .$$

$z(\alpha)$ křivka v M s teč. vekt a

pull-back (stáhnutí) 1-formy



$\phi^* : \mathbb{T}^*_{\phi x} N \to \mathbb{T}^*_x M$

$$(\phi^* \tilde{\omega}) \cdot a|_x = \tilde{\omega} \cdot (\phi_* a)|_{\phi x}$$

platí

$$(\phi_* a)|_{\phi x} [\hat{f}] = a|_x [\phi^* \hat{f}]$$

$(\phi_* a)|_{\phi x} [\hat{f}] = \frac{d}{d\alpha} \hat{f}(\phi z(\alpha))\big|_{\alpha=0} = \frac{d}{d\alpha} (\phi^* \hat{f})(z(\alpha))\big|_{\alpha=0} = a|_x [\phi^* \hat{f}] \qquad a = \dot{z}$

$$\phi^*(d\hat{f}|_{\phi x}) = d(\phi^* \hat{f})|_x$$

$a|_x \cdot \phi^*(d\hat{f}|_{\phi x}) = (\phi_* a)|_{\phi x} \cdot d\hat{f}|_{\phi x} = (\phi_* a)|_{\phi x} [\hat{f}] = a|_x [\phi^* \hat{f}] = a|_x \cdot d(\phi^* \hat{f})|_x$

$\Rightarrow \phi^*(d\hat{f}|_{\phi x}) = d(\phi^* \hat{f})|_x$

linearita indukovaných zobr.

$$\phi_*(a+rb) = \phi_* a + r\phi_* b \qquad r\in\mathbb{R}$$

$$\Leftarrow (\phi_*(a+rb))[\hat{f}] = (a+rb)[\phi^*\hat{f}] = a[\phi^*\hat{f}] + r\, b[\phi^*\hat{f}] = (\phi_* a)[\hat{f}] + r(\phi_* b)[\hat{f}]$$

$$\phi^*(\tilde{\alpha} + r\tilde{\beta}) = \phi^*\tilde{\alpha} + r\,\phi^*\tilde{\beta}$$

$$\Leftarrow \phi^*(\tilde{\alpha}+r\tilde{\beta})\cdot u = (\tilde{\alpha}+r\tilde{\beta})\cdot\phi_* u = \tilde{\alpha}\cdot\phi_* u + r\tilde{\beta}\cdot\phi_* u = (\phi^*\tilde{\alpha} + r\phi^*\tilde{\beta})\cdot u$$

zobecnění na tenzory

komutace s tenz. násobením $\otimes$ → rozšíření

$$\phi_* : \mathbf{T}_x{}^p_0 M \to \mathbf{T}_{\phi x}{}^p_0 N$$

$$\phi^* : \mathbf{T}_{\phi x}{}^0_p N \to \mathbf{T}_x{}^0_p M$$

nelze definovat na obecných tenz $\mathbf{T}^p_q$ pro obecné ϕ

indukt. zobr. pro difeomorf.

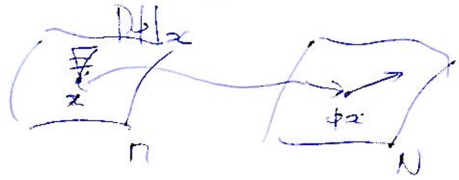
podmínka $\phi^* = \phi^{-1}{}_*$ $\quad \phi_* = \phi^{-1*}$ umožňuje rozšíření

$$\phi_* : \mathbf{T}_x{}^p_q M \to \mathbf{T}_{\phi x}{}^p_q N$$

$$\phi^* : \mathbf{T}_{\phi x}{}^p_q N \to \mathbf{T}_x{}^p_q M$$

Diferenciál zobrazení

díky linearitě indukovaného zobr. lze repr. tenzorově



$$D\phi|_x \in T_x^* M \otimes T_{\phi x} N$$

$$\underset{T_{\phi x} N}{(\phi_* a)^{\tilde{n}}} = D^{\tilde{n}}_{m}\phi|_x \, \underset{T_x M}{a^m}$$

$$\underset{T_x^* M}{(\phi^* \tilde{\alpha})_m} = D^{\tilde{n}}_{m}\phi|_x \, \underset{T^*_{\phi x} N}{\tilde{\alpha}_{\tilde{n}}}$$

vyjádření v souřadnicích

(U, x) mapa na M $\quad$ (V, y) mapa na N

$$\Phi = y \circ \phi \circ x^{-1}$$

$$D\phi = \frac{\partial \Phi^{\tilde{b}}}{\partial x^a} \, dx^a \, \frac{\partial}{\partial y^{\tilde{b}}}$$

plyne z aplikace $\phi_* \frac{\partial}{\partial x^a}$ a $\phi^* dy^{\tilde{b}}$

složení zobrazení

$$(\phi \circ \psi)_* = \phi_* \circ \psi_* \quad \Rightarrow \quad D(\phi \circ \psi)|_x = D\phi|_{\psi x} \cdot D\psi|_x$$

působení na tenzorech

ϕ difeomorf.

$$\phi^{-1} \circ \phi = \mathrm{id}_M \quad \phi \circ \phi^{-1} = \mathrm{id}_N \quad \Rightarrow \quad D\phi \cdot D\phi^{-1} = \delta_N \quad D\phi^{-1} \cdot D\phi = \delta_M$$

$$(\phi_* T)^{\tilde{a}_1 \tilde{a}_2 \dots}_{\tilde{b}_1 \tilde{b}_2 \dots} = D^{\tilde{a}_1}_{m_1}\phi \, D^{\tilde{a}_2}_{m_2}\phi \dots D^{n_1}_{\tilde{b}_1}\phi^{-1} \, D^{n_2}_{\tilde{b}_2}\phi^{-1} \dots T^{m_1 m_2 \dots}_{n_1 n_2 \dots}$$

speciální případy

1-dim varieta M lze ztotožnit s $\mathbb{R}$
volbou souř. τ

TM je 1. dim — ztotož. s $\mathbb{R}$ pomocí $\frac{\partial}{\partial\tau}$
T^*M je 1 dim. — ztotož. s $\mathbb{R}$ pomocí $d\tau$

funkce na M

$f: M \to N = \mathbb{R}$ $\qquad \bar{f} = \tau(f) : M \to \mathbb{R}$

$Df = d\bar{f}\,\frac{\partial}{\partial\tau} = \bar{f}_{,i}\, dx^i \frac{\partial}{\partial\tau}$

lze ztotožnit Df s $d\bar{f}$

křivka v N

$z : M = \mathbb{R} \to N$ $\qquad \bar{z} \circ \tau = z$ $\qquad \bar{z} : \mathbb{R} \to N$

$Dz = d\tau \frac{D\bar{z}}{d\tau} = \frac{d\bar{z}^i}{d\tau} d\tau \frac{\partial}{\partial y^i}$ $\qquad \bar{z}^i = y^i(\bar{z})$

lze ztotožnit Dz a $\frac{D\bar{z}}{d\tau} \equiv \dot{\bar{z}}$

Indukované zobrazení na polích

funkce $\quad \phi^*: \mathcal{F}N \to \mathcal{F}M$

$$(\phi^* \tilde{f})(x) = \tilde{f}(\phi x)$$

formy $\quad \phi^* \quad \mathcal{T}^0_p N \to \mathcal{T}^0_p M$

$$\underbrace{(\phi^* \tilde{\omega})(x)}_{\mathcal{T}_x{}^0_p M} = \phi^*\big(\underbrace{\tilde{\omega}(\phi x)}_{\mathcal{T}_{\phi x}{}^0_p N}\big)$$

platí

$$\phi^* df = d\phi^* f$$

plyne z relace v bodě

nelze definovat push-forward vekt pole

$$\underbrace{(\phi_* a)(\tilde{x})}_{\mathcal{T}_{\tilde{x}} N} = \phi_*\big(\underbrace{a(\phi^{-1}\tilde{x})}_{\mathcal{T}_{\phi^{-1}\tilde{x}} M}\big) \qquad \phi^{-1} \text{ není definovaný!!}$$

potřeba inverze ϕ

difeomorfismy

push-forward:

$$(\phi_* A)(\tilde{x}) = \phi_*\big(A(\phi^{-1}\tilde{x})\big)$$
$$(\phi_* A)(\phi x) = \phi_*(A(x))$$

pullback

$$(\phi^* \tilde{A}) = \phi_*^{-1} A$$
$$(\phi^* \tilde{A})(x) = \phi^* \tilde{A}(\phi x)$$

konzistentní s definicí ϕ^* pro formy

vlastnosti

$$\phi_*(A + \alpha B) = \phi_* A + \alpha\, \phi_* B$$
$$\phi_*(A \otimes B) = (\phi_* A) \otimes \phi_* B$$
$$\phi_*\, \mathsf{C} A = \mathsf{C}\, \phi_* A$$

lastnosti

$$a[\phi^* \tilde{f}] = \phi^*\big((\phi_* a)[\hat{f}]\big)$$

$$\phi_*\big(a[\phi^* \hat{f}]\big) = (\phi_* a)[\hat{f}]$$

$$\phi_*\big(a[f]\big) = \phi_* a[\phi_* f] \qquad \text{ekvivalentni}$$

dokaz

$$(a[\phi^*\hat{f}])(x) = a(x)[\phi^*\hat{f}] = \big(\phi_*(a(x))\big)[\tilde{f}] \overset{\text{difeo}}{=}$$

$$= \big((\phi_* a)(\phi x)\big)[\hat{f}] = \big((\phi_* a)[\hat{f}]\big)\big|_{\phi x} = \phi^*\big((\phi_* a)[\hat{f}]\big)(x)$$

Lieove zavorke

$$\phi_*[a,b] = [\phi_* a, \phi_* b]$$

dokaz

$$\phi^*\big((\phi_*[a,b])[\hat{f}]\big) = [a,b][\phi^*\hat{f}] = a[b[\phi^*\hat{f}]] - b[a[\phi^* \hat{f}]]$$

$$= a[\phi^*((\phi_* b)[\hat{f}])] - b[\phi^*((\phi_* a)[\hat{f}])]$$

$$= \phi^*\big(\phi_* a[(\phi_* b)[\hat{f}]] - (\phi_* b)[(\phi_* a)[\hat{f}]]\big)$$

$$= \phi^*\big([\phi_* a, \phi_* b][\hat{f}]\big)$$

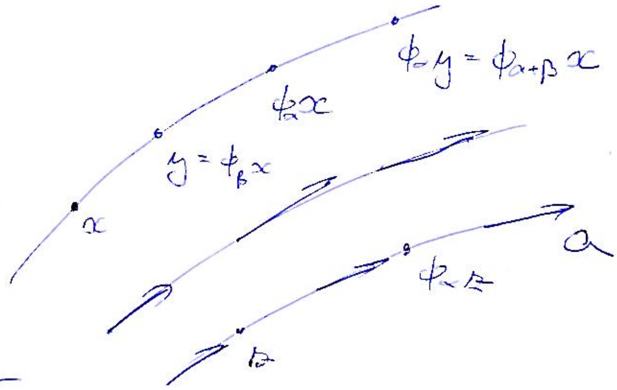
Tok a jeho generátor

Def tok na M, tzv. 1-param. grupa diffeo

$\phi_\alpha \in \mathrm{Diff}\, M \qquad \alpha \in \mathbb{R}$

$\phi_\alpha \circ \phi_\beta = \phi_{\alpha+\beta}$

$\phi_0 = id \qquad \phi_{-\alpha} = \phi_\alpha^{-1}$



všechny body na orbitě $\phi_\alpha x$ se pohybují po stejné orbitě

Def generátor toku

a je generátor ϕ_α $\equiv$

$$a|_x = \frac{D}{d\alpha} \phi_\alpha x \Big|_{\alpha=0} \qquad \phi_\alpha = \mathrm{diff}_\alpha[a]$$

Tk: tok a jeho generátor se vzájemně jedn. určují

tj gen a definuje jedn. ϕ_α

$\Leftarrow$ věty o řešení dif. rov. 1. řádu

Pozn:

$\phi_{\alpha*} a = a$

$$\Leftarrow (\phi_{\alpha*} a)(\phi_\alpha x) = \phi_{\alpha*}(a(x)) = \phi_{\alpha*}\left(\frac{D}{d\varepsilon} \phi_\varepsilon x \Big|_{\varepsilon=0}\right) = \frac{D}{d\varepsilon} \phi_\varepsilon \phi_\alpha x \Big|_{\varepsilon=0} = a(\phi_\alpha x)$$

Lieova derivace

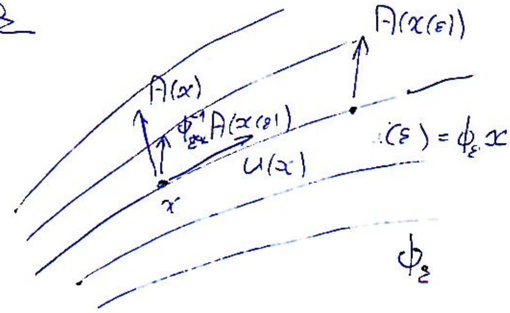
chceme charakteriz. derivaci tenz pole A ve směru u
formální výraz

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(A(x+\varepsilon u)-A(x)\right)$$

malorozměrné křivka ve směru u

$T_{x(\varepsilon)}M$ $\quad$ T_xM – nelze porovnávat

potřeba přenést tenzory do stejného bodu
nelze provést pouze pomocí specifikace směru $u\in T_xM$
potřeba dodatečné struktury – tok



Def: Lieova derivace

$$\mathcal{L}_u A\big|_x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\phi_{-\varepsilon *}(A(\phi_\varepsilon x)) - A(x)\right)$$

kde ϕ_α je tok generovaný u

alternativní zápisy

$$\mathcal{L}_u A\big|_x = \frac{d}{d\varepsilon}\phi_{-\varepsilon *}(A(\phi_\varepsilon x))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\phi^*_\varepsilon(A(\phi_\varepsilon x))\Big|_{\varepsilon=0}$$

$$\mathcal{L}_u A = -\frac{d}{d\varepsilon}\phi_{\varepsilon *}A\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\phi^*_\varepsilon A\Big|_{\varepsilon=0}$$

$$\Leftarrow \phi_{-\varepsilon *}(A(\phi_\varepsilon x)) = (\phi_{-\varepsilon *}A)(x) = (\phi^*_\varepsilon A)(\alpha)$$

Th. ϕ_α tok s generátorem a
indukované zobr ϕ^* na tenz. polích splňuje

$$\phi^*_0 = id \qquad \phi^*_\alpha\circ\phi^*_\beta = \phi^*_{\alpha+\beta} \qquad \frac{d}{d\alpha}\phi^*_\alpha\Big|_{\alpha=0} = \mathcal{L}_a$$

proto splňuje i

$$\frac{d}{d\alpha}\phi^*_\alpha = \phi^*_\alpha\mathcal{L}_a \quad\Rightarrow\quad \phi^*_\alpha = \exp[\alpha\mathcal{L}_a]$$

důk:

$$\phi^*_\alpha\mathcal{L}_a A = \phi^*_\alpha\frac{d}{d\varepsilon}\phi^*_\varepsilon A\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\phi^*_{\alpha+\varepsilon}A\Big|_{\varepsilon=0} = \frac{d}{d\alpha}\phi^*_\alpha A$$

vlastnosti Lieovy derivace

$$\mathcal{L}_u(A+\alpha B) = \mathcal{L}_u A + \alpha \mathcal{L}_u B \qquad \alpha \in \mathbb{R}$$

$$\mathcal{L}_u(A\otimes B) = (\mathcal{L}_u A)\otimes B + A\otimes(\mathcal{L}_u B)$$

$$\mathcal{L}_u \mathsf{C} A = \mathsf{C}\, \mathcal{L}_u A$$

$$\mathcal{L}_u f = u\cdot df$$

$$\mathcal{L}_u df = d\mathcal{L}_u f$$

$$\mathcal{L}_u(a\cdot\omega) = (\mathcal{L}_u a)\cdot\omega + a\cdot\mathcal{L}_u\omega$$

$$\mathcal{L}_u v = [u,v]$$

důkazy

převážně plyne z vlastností ϕ^*

– komutace s $+, \otimes, \mathsf{C}, d$

a převedení na obyčejnou der. $\frac{d}{d\varepsilon}\phi_\varepsilon^* A\big|_{\varepsilon=0}$

$\mathcal{L}_a f = a[f]$ – přímo z def.

$$\mathcal{L}_u(v\cdot df) = u\cdot d(v\cdot df) =$$

$$= (\mathcal{L}_u v)\cdot df + v\cdot\mathcal{L}_a df = (\mathcal{L}_u v)\cdot df + \cancel{v}\cdot d(u\cdot df)$$

$$\Rightarrow (\mathcal{L}_u v)\cdot df = u[v[f]] - v[u[f]] = [u,v][f]$$

Th: Lieova der. $\mathcal{L}_a$ je tenz. derivace

na FM určena vekt. polem a

na TM určena Lieovou závorkou s a

!! jednoznačně fixována na tenz. polích

Tl: linearita (nad $\mathbb{R}$) ve směru

$$\mathcal{L}_{u+\alpha v} = \mathcal{L}_u + \alpha \mathcal{L}_v \qquad \alpha \in \mathbb{R}$$

důk:

$\mathbb{L} = \mathcal{L}_{u+\alpha v} - \mathcal{L}_u - \alpha \mathcal{L}_v$ je tenz. der. splňující

$\mathbb{L} f = (u+\alpha v)[f] - u[f] - \alpha v[f] = 0 \qquad \mathbb{L} w = [(u+\alpha v), w] - [u,w] - \alpha [v,w] = 0$

$\Rightarrow \mathbb{L} = 0$ c.b.d.

Tl:

$$\mathcal{L}_{[u,v]} = \mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u$$

důk:

$\mathbb{L} = \mathcal{L}_{[u,v]} - (\mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u)$ je tenz. der.

linearita zřejmá

Leibnitz

$[\mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u](AB) = \mathcal{L}_u((\mathcal{L}_v A)B + A \mathcal{L}_v B) - (u \leftrightarrow v)$

$= (\mathcal{L}_u \mathcal{L}_v A)B + A(\mathcal{L}_u \mathcal{L}_v B) + (\mathcal{L}_v A)(\mathcal{L}_u B) + (\mathcal{L}_u A)(\mathcal{L}_v B) - (u \leftrightarrow v)$

$= ([\mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u] A) B + A([\mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u] B)$

$\Rightarrow \mathbb{L}(AB) = (\mathbb{L} A) B + A (\mathbb{L} B)$

na funkcích $\quad \mathbb{L} f = [u,v][f] - (u[v[f]] - v[u[f]]) = 0$

na vekt. $\quad \mathbb{L} w = [[u,v],w] - [u,[v,w]] - [v[u,w]] = 0 \quad$ J.I.

$\Rightarrow \mathbb{L} = 0$ c.b.d.

Tl: $\mathcal{L}_w [u,v] = [\mathcal{L}_w u, v] + [u, \mathcal{L}_w v]$

důk:

$[w,[u,v]] = [[w,u],v] + [u,[w,v]] \quad \Leftarrow$ J.I. c.b.d.

Tl: souřadnicové vyjádření

$$\left(\mathcal{L}_{\frac{\partial}{\partial x^1}} A\right)^{a\dots}_{b\dots} = A^{a\dots}_{b\dots,1} \qquad \text{kde } A^{a\dots}_{b\dots} \text{ jsou kom. nič } x^a$$

důk:

$\mathcal{L}_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^a} = 0 \qquad \mathcal{L}_{\frac{\partial}{\partial x^1}} dx^a = d \frac{\partial x^a}{\partial x^1} = 0 \qquad \mathcal{L}_{\frac{\partial}{\partial x^1}} (A^{a\dots}_{b\dots}) = A^{a\dots}_{b\dots,1}$

Interpretace Lieovy závorky

Tk: ϕ_α, ψ_β toky s generátory a, b, platí

$$\phi_\alpha \psi_\beta = \psi_\beta \phi_\alpha \iff [a,b]=0$$

důk:

$\Rightarrow$ definujme $\Xi(\alpha,\beta) = \phi_\alpha \psi_\beta$

orbita $\Xi(\alpha,\beta)x_0$ vytváří 2-di. varietu N_{x_0} se souř. α, β

platí $a|_{N_{x_0}} = \frac{\partial}{\partial\alpha}$ $\quad b|_{N_{x_0}} = \frac{\partial}{\partial\beta}$ $\Rightarrow [a,b]|_{N_{x_0}} = [\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}] = 0$ –

různou volbou x_0 se dostane foliace M podvar. N_x $\Rightarrow [a,b]=0$

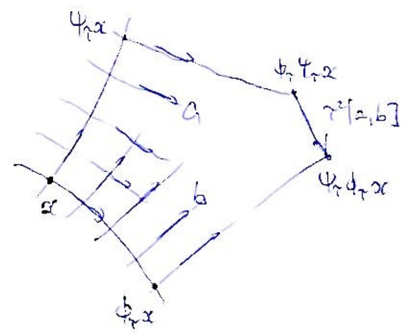
$\Leftarrow$ $\phi_\alpha^* = \exp[\alpha\mathcal{L}_a]$ $\quad \psi_\beta = \exp[\beta\mathcal{L}_b]$

$[a,b]=0 \Rightarrow [\mathcal{L}_a, \mathcal{L}_b]=0 \Rightarrow \exp[\alpha\mathcal{L}_a]\exp[\beta\mathcal{L}_b] = \exp[\beta\mathcal{L}_b]\exp[\alpha\mathcal{L}_a] \Rightarrow \phi_\alpha\psi_\beta = \psi_\beta\phi_\alpha$

Tk: ϕ_α, ψ_β toky s generátory a, b

$$[\phi_\tau, \psi_\tau]^* f = \tau^2 [a,b]\cdot df + O(\tau^3)$$

důkaz %



Tk $a_k \in TU$ $\quad k=1\dots K$ $\quad U$ okolí x

$[a_k, a_l] = 0$ na U $\quad a_k$ nezávislé

$\Rightarrow$ bodem x_0 prochází podvarieta N_{x_0} ke které jsou a_k tečné a na ní lze zvolit souř. X^k tak, že $a_k = \frac{\partial}{\partial X^k}$

důk – viz 2D případ

${}^k\phi_\alpha$ tok generovaný a_k

$\Xi(\alpha^1,\dots,\alpha^K) = {}^1\phi_{\alpha^1}\cdots{}^K\phi_{\alpha^K}$ nezávisí na pořadí ${}^k\phi_{\alpha^k}$

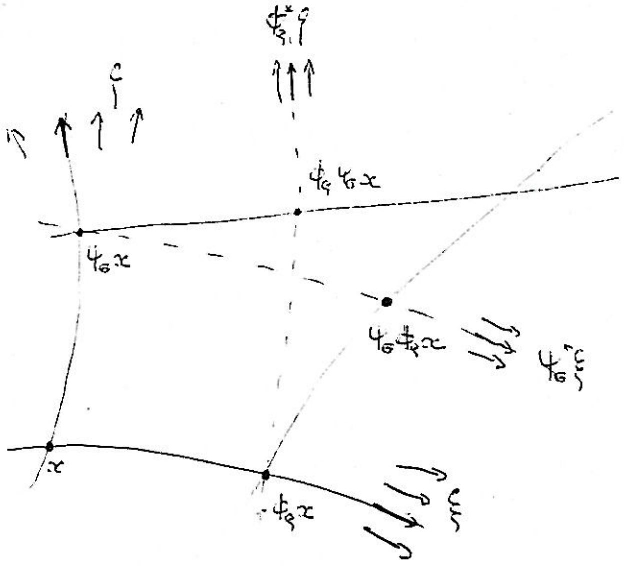
$N_x = \Xi(\alpha^1,\dots,\alpha^K)x_0$ $\quad \alpha^k \in (-\varepsilon,\varepsilon)$ $\quad$ – K-di podvarieta M

$X^k(\Xi(\alpha^a)x_0) = \alpha^k$ $\rightarrow$ souř. $X = [X^k]$

zřejmě $\frac{\partial}{\partial X^k}\Big|_x = \frac{D}{d\alpha^k}\Xi(\alpha^1\dots\alpha^k\dots\alpha^K)x_0\Big|_{\substack{\alpha^k = X^k \\ X^j = X^j(x)}} = a_k|_x$

důsledek

e_k je holonomní báze $\iff [e_k, e_l] = 0$



$$\varphi(\varrho,\sigma) = f(\varphi_\varrho \psi_\sigma x) \qquad \varphi(0,0) = f(x)$$

$$\psi(\sigma,\varrho) = f(\psi_\sigma \varphi_\varrho x) \qquad \psi(0,0) = f(x)$$

$$\frac{\partial \varphi}{\partial \varrho}(\varrho,\sigma) = (\xi \cdot df)\big|_{\varphi_\varrho \psi_\sigma x} \qquad \frac{\partial \varphi}{\partial \varrho}(0,0) = \xi \cdot df\big|_x$$

$$\frac{\partial \varphi}{\partial \sigma}(\varrho,\sigma) = ((\varphi^*_\varrho \zeta)\cdot df)\big|_{\varphi_\varrho \psi_\sigma x} \qquad \frac{\partial \varphi}{\partial \sigma}(0,0) = \zeta \cdot df\big|_x$$

$$\frac{\partial \psi}{\partial \sigma}(\sigma,\varrho) = (\zeta \cdot df)\big|_{\psi_\sigma \varphi_\varrho x} \qquad \frac{\partial \psi}{\partial \sigma}(0,0) = \zeta \cdot df\big|_x$$

$$\frac{\partial \psi}{\partial \varrho}(\sigma,\varrho) = ((\psi^*_\sigma \xi)\cdot df)\big|_{\psi_\sigma \varphi_\varrho x} \qquad \frac{\partial \psi}{\partial \varrho}(0,0) = \xi \cdot df\big|_x$$

$$\frac{\partial^2 \varphi}{\partial \varrho^2}(\varrho,\sigma) = (\xi \cdot d(\xi \cdot df))\big|_{\varphi_\varrho \psi_\sigma x} \qquad \frac{\partial^2 \varphi}{\partial \varrho^2}(0,0) = \xi \cdot d(\xi \cdot df)\big|_x$$

$$\frac{\partial^2 \varphi}{\partial \sigma^2}(\varrho,\sigma) = ((\varphi^*_\varrho \zeta)\cdot d((\varphi^*_\varrho \zeta)\cdot df))\big|_{\varphi_\varrho \psi_\sigma x} \qquad \frac{\partial^2 \varphi}{\partial \sigma^2}(0,0) = \zeta \cdot d(\zeta \cdot df)\big|_x$$

$$\frac{\partial^2 \varphi}{\partial \sigma \partial \varrho}(\varrho,\sigma) = ((\varphi^*_\varrho \zeta)\cdot d(\xi \cdot df))\big|_{\varphi_\varrho \psi_\sigma x} \qquad \frac{\partial^2 \varphi}{\partial \sigma \partial \varrho}(0,0) = \zeta \cdot d(\xi \cdot df)\big|_x$$

$$\frac{\partial^2 \psi}{\partial \sigma^2}(\sigma,\varrho) = (\zeta \cdot d(\zeta \cdot df))\big|_{\psi_\sigma \varphi_\varrho x} \qquad \frac{\partial^2 \psi}{\partial \sigma^2}(0,0) = \zeta \cdot d(\zeta \cdot df)\big|_x$$

$$\frac{\partial^2 \psi}{\partial \varrho^2}(\sigma,\varrho) = ((\psi^*_\sigma \xi)\cdot d((\psi^*_\sigma \xi)\cdot df))\big|_{\psi_\sigma \varphi_\varrho x} \qquad \frac{\partial^2 \psi}{\partial \varrho^2}(0,0) = \xi \cdot d(\xi \cdot df)\big|_x$$

$$\frac{\partial^2 \psi}{\partial \varrho \partial \sigma}(\sigma,\varrho) = ((\psi^*_\sigma \xi)\cdot d(\zeta \cdot df))\big|_{\psi_\sigma \varphi_\varrho x} \qquad \frac{\partial^2 \psi}{\partial \varrho \partial \sigma}(0,0) = \xi \cdot d(\zeta \cdot df)\big|_x$$

$$f(\psi_\tau \varphi_\tau x) - f(\varphi_\tau \psi_\tau x) = \psi(\tau,\tau) - \varphi(\tau,\tau) =$$

$$= \left(\psi - \varphi + \left(\frac{\partial \psi}{\partial \sigma} + \frac{\partial \psi}{\partial \varrho} - \frac{\partial \varphi}{\partial \varrho} - \frac{\partial \varphi}{\partial \sigma}\right)\tau + \left(\frac{1}{2}\frac{\partial^2 \psi}{\partial \sigma^2} + \frac{1}{2}\frac{\partial^2 \psi}{\partial \varrho^2} + \frac{\partial^2 \psi}{\partial \varrho \partial \sigma} - \frac{1}{2}\frac{\partial^2 \varphi}{\partial \varrho^2} - \frac{1}{2}\frac{\partial^2 \varphi}{\partial \sigma^2} - \frac{\partial^2 \varphi}{\partial \sigma \partial \varrho}\right)\tau^2 + \dots\right)\Big|_{0,0}$$

$$= \tau^2\left(\xi \cdot d(\zeta \cdot df) - \zeta \cdot d(\xi \cdot df)\right)\big|_x =$$

$$= \tau^2[\xi,\zeta]\cdot df\,\big|_x$$